DOCUMENT RESUME

ED 170 330 TM 008 550

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TITLE The Indeterminacy of Common Factors.
PUB DATE [78]
NOTE 13p.; Paper presented at the Annual Meeting of the American Psychological Association (86th, Toronto, Ontario, Canada, August 28-September 1, 1978)

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *Cluster Analysis; Correlation; *Factor Analysis; *Predictor Variables; *Research Problems; *Statistical Studies
IDENTIFIERS *Factor Indeterminacy

ABSTRACT

If the factor loadings of a core set of variables (fitting the general factor model) are the same when analyzed alone, and when analyzed along with the rest of the variables in an infinite behavior domain, there is only one factor variable in the domain that is a possible factor variable of the core set. If the condition of consistency of factor loadings is not satisfied, there may exist subdomains whose factor variables are possible factors of the core set. This observation serves to provide empirical significance for the mathematical problem of factor indeterminacy. (Author/GDC)

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THE INDETERMINACY OF COMMON FACTORS

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Abstract

It is shown that if the factor loadings of a core set of variables (fitting the general factor model) are the same when it is analyzed alone, and when it is analyzed along with the rest of the variables in an infinite behavior domain, there is only one factor variable in the domain that is a possible factor variable of the core set.

If the condition of consistency of factor loadings is not satisfied, there may exist subdomains whose factor variables are possible factors of the core set.

This observation serves to provide empirical significance for the mathematical problem of factor indeterminacy.

It is widely known that the common factor model is believed to have special problems arising from the indeterminacy of its factor scores. From the relatively recent papers (Guttman, 1955, 1956, 1957; Schönemann, 1971; Schönemann and Wang, 1972; McDonald, 1974; Mulaik, 1972, 1976; Green, 1976) on this topic, it is clear that it is regarded as both important and controversial. Practising factor analysts not directly acquainted with the question seem either to ignore the issue completely, or to accept on authority that factor indeterminacy is a fault peculiar to the common factor model, which justifies avoiding it, and substituting an approximately equivalent model which has not been declared to have an equivalent fault. One reason why the factor score indeterminacy issue is easy to ignore is that most work on the subject has failed to exhibit a clear logical connection between the indeterminacy of factor scores and the practice of factor analysis.

In this paper we give a relatively nonmathematical account of factor score indeterminacy, and its implications for the definition of common factors, and for the practice of factor analytic research. We simplify the task by considering only the special case in which we have just one general factor (the Spearman case).

It is well known that if a finite number of variables (tests, say) fit the general factor model, and no test is perfectly correlated with the factor, then the multiple correlation of the common factor with the tests is strictly

less than unity. If we can conceivably find infinitely many variables with nonzero loadings on a factor, then in the limit as the number of variables approaches infinity this multiple correlation approaches unity. The fact that the general factor cannot be determined with perfect reliability by a test battery of finite length is not, however, the problem of factor score indeterminacy. Clearly we expect all social science measurements to be imperfect, and we can certainly obtain multiple correlations of tests with a factor in the general range of acceptable reliability coefficients.

Rather, the problem of factor score indeterminacy is as follows. A necessary and sufficient condition that a variable has the properties of a general factor of a set of tests is that its correlations with the tests are the same as their factor loadings on the factor. If a variable has these correlations with the tests, we say that it is a possible factor variable of those tests. In general there is more than one such possible factor variable.

If ρ^2 is the squared multiple correlation of the general factor with n given tests, $Y_1, \ldots, Y_n$, and X_1 and X_2 are two possible factor variables, then we can obtain bounds on the correlation ρ_{12} between X_1 and X_2 by showing that their partial correlation with the n tests "partialled out" is given by $(\rho_{12} - \rho^2)/(1 - \rho^2)$. It follows that $2\rho^2 - 1 \leq \rho_{12} \leq 1$. This result was first given by Guttman (1955).

That there is a problem of factor score indeterminacy beyond mere imprecision of estimation, is first noticed when we observe that if the multiple correlation between the tests and a possible factor variable is less than .707, the correlation between two such possible factor variables can be as low as zero. It is not immediately obvious what empirical interpretation to place on this fact, however.

Guttman (1955) showed that we may construct alternative factor variables, in infinitely many ways, by adding to the regression estimate a value of a random variable defined "by throwing dice or turning a roulette wheel" and claimed that this fact renders suspect the practice of naming factors by studying factor loadings. More precisely, to construct a possible factor score for any person randomly drawn from a defined population, we make one throw of a die, or turn of a roulette wheel (or generate one number with a random number generator), and arbitrarily associate the number so obtained with the scores of the randomly drawn person. Schönemann and Wang (1972) suggested similar procedures for constructing possible factor scores, remarking that since infinitely many different sets of such numbers can be computed, they need not be estimated.

However, while such computational procedures do indeed yield mathematically admissible alternative factor variables, the solutions so obtained can hardly be regarded as measurements of empirical properties of the persons in the population, or as lending themselves to distinct

interpretations. Hence the argument that factor score indeterminacy renders the interpretation of factor loadings suspect is thus far incomplete.

One way to give empirical meaning to alternative possible common factor variables has been shown by Mulaik and McDonald (in press). They consider the possible factors of two sets of variables obtained by augmenting a core set of n marker variables for a factor, with different sets of additional variables, subject only to the condition that each augmented set shall continue to satisfy a common factor model, consistent with the model for the core set. They discuss what we might call alternative tests of infinite length, extensions of the core set, each uniquely determining a factor variable. They give conditions under which these tests of infinite length will determine the same factor variable.

Here we develop a simpler treatment of the problem, based directly on behavior domain theory. Following Guttman (1957) we suppose we have n variables $Y_1, \ldots, Y_n$ drawn from a set of $n+m$ empirical variables $Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+m}$, a universe of content or behavior domain that is defined in advance of any statistical analysis.

To establish any relationship between the possible factor variables of $Y_1, \ldots, Y_n$, and those of the entire behavior domain, we require the condition that there is a joint factor analysis of the $n+m$ variables in which the factor loadings of the first n are the same as when these

are factored without the further m variables. If this condition is satisfied, we say that the core and the domain have consistent loadings. The following fundamental result is easily proved: A possible common factor variable of the domain is a possible general factor of the core factored separately, if and only if the domain and the core have consistent loadings. Further, if m, the number of additional variables in the domain, is infinitely large, then its possible factor variable (marked by the core set) is uniquely determined by the (n + m) variables of the domain, since its multiple correlation with them is unity.

Thus, if the joint factor pattern is consistent with the core factor pattern, so that possible factor variables of the domain and the core may be defined on the same basis, then if the additional variables of the domain have nonzero loadings on the factor that is marked by the core, there is one and only one factor variable in the domain that is a possible factor of the core set.

It is important to recognize that there is no reason to suppose that the condition of consistent loadings will always, or even usually, be fulfilled in practice. It may be shown to correspond to a generalized counterpart of the condition that there shall be no Heywood variables. We know that the condition has been violated if we attempt to add variables to a core set, prescribing a consistent analysis, and we obtain a Heywood case. But generally we

will not know whether or not the domain contains variables which, if they were added to the core, would violate the required condition.

Mulaik and McDonald (in press) have shown that if the conditions for consistency of a domain and a core set are not satisfied, the domain can possibly be divided into overlapping subdomains, all containing the core set as a subset, and possessing disjoint sets of factor variables with possible correlations that are only bounded as are the correlations between alternative factors of the core set. As a consequence, if two investigators take a common core of variables as markers for a factor, and independently draw further variables from such a domain, not by a sampling rule, but merely augmenting the core set subject to the satisfaction of the general factor model, they may define two distinct tests of infinite length, containing distinct variables which lend themselves to distinct interpretations. Only when the two sets of augmenting variables were merged would we discover that the domain does not have a basis consistent with the core, and hence has no factor variable that is a possible factor of the core.

Thus, these observations do not define away the problem of factor score indeterminacy. Rather, they point to its significance for the user of the model. Few if any investigators base their factor analyses on tests obtained by a clear rule for selecting tests representative of a precisely defined behavior domain. The assumption that the

behavior domain will yield consistent factor loadings for the observed (core) variables is a very strong, falsifiable hypothesis, which we seldom attempt to falsify. If there is a behavior domain, defined in advance, it is at least possible to falsify the hypothesis. In the absence of an agreed behavior domain, two investigators may form quite incompatible extended tests from a common core of marker variables, and never discover the incompatibility. It remains for appropriate empirical work, using merged variables, augmented sets, and extension procedures, say, to establish how serious these problems are in practice.

However, it may be pointed out that these problems of the common factor model do not, as is sometimes supposed, constitute grounds for using alternative modes of analysis such as component theory. McDonald (1975) has discussed some aspects of this choice. Here we recognize that if we worry about the relationship of factor score estimates to the variables being estimated, it is because we are able, under certain conditions, to define the variables being estimated on the basis of a behavior domain that fits the common factor model consistently with the given variables. To show that component scores do not have an indeterminacy problem, as studied here, we must show that the component score of a core set of variables uniquely determines a "corresponding" component score in an infinite domain from which the core set has been drawn. But we know immediately that this cannot be true. Thus it would seem that

mathematical arguments (Schönemann and Steiger, 1976, for example) favorably contrasting component theories with common factor theory, cannot be accepted until they are reformulated in terms of behavior domains, and the relation of core components to domain components. It seems unlikely that they will yield better representations of components of interest in a behavior domain than we have in common factor estimates. (See McDonald, 1977, for a more extended account of this question.)

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